

# $S$ -Duality and the Dyon Spectrum in $N = 2$ Super Yang-Mills Theory

Jerome P. Gauntlett

*Lauritsen Lab  
California Institute of Technology  
Pasadena, CA 91125  
jerome@theory.caltech.edu*

Jeffrey A. Harvey

*Enrico Fermi Institute, University of Chicago  
5640 Ellis Avenue, Chicago, IL 60637  
harvey@poincare.uchicago.edu*

## Abstract

We study the dyon spectrum in  $N = 2$  super Yang-Mills theory with gauge group  $SU(2)$  coupled to  $N_f$  matter multiplets in the fundamental representation. For magnetic charge one and two we determine the spectrum explicitly and show that it is in agreement with the duality predictions of Seiberg and Witten. We briefly discuss the extension to higher charge monopoles for the self-dual  $N_f = 4$  case and argue that the conjectured spectrum of dyons predicts the existence of certain harmonic spinors on the moduli space of higher charge monopoles.

## 1. Introduction

There has been impressive progress recently in understanding the dynamics of  $N = 2$  supersymmetric gauge theories [1,2]. This progress has relied on the powerful constraints which come from the holomorphic structure of supersymmetric gauge theories and on a duality between electric and magnetic degrees of freedom [3]. This duality symmetry is far from apparent in the current formulation of gauge theory and is still rather poorly understood.

It has also become clear that there are theories which may possess an exact duality, relating correlation functions in the dual theories at all scales, and other theories which possess only effective dualities in the infrared limit. It may be that a full understanding of theories which possess an exact duality will lead to a better understanding of theories with effective duality. One class of theories that are thought to possess an exact duality are the  $N = 4$  supersymmetric Yang-Mills theories [4,5]. These theories have perturbatively vanishing beta functions. In [2] it was conjectured that the simplest  $N = 2$  gauge theory with perturbatively vanishing beta function i.e. gauge group  $G = SU(2)$  and  $N_f = 4$  hypermultiplets in the fundamental representation, may also have an exact  $SL(2, Z)$  electric-magnetic  $S$ -duality<sup>1</sup>. The arguments of [2] were based on an analysis of the dynamics of the theory and its relation to the dynamics of theories with  $N_f < 4$ .

It was further noted in [2] that  $S$ -duality also makes predictions about the spectrum of BPS dyon states in the  $N_f = 4$  theory. These are states whose masses saturate a Bogomol'nyi bound depending on their electric and magnetic charges. In this paper we will analyze the spectrum of BPS dyons using semi-classical techniques and translate the conjectured spectrum into predictions about the existence of certain harmonic forms on the moduli space of BPS monopole solutions. We will verify these predictions in the one and two-monopole sectors of the theory. The analysis is similar in spirit to that of Sen [5] who made and verified the analogous prediction in the context of  $N = 4$  super Yang-Mills theory. Our analysis for  $N_f = 4$  also verifies, *en passant*, a prediction made in [2] for the  $N_f = 3$  theory concerning the existence of a dyon state with magnetic charge two.

The outline of this paper is as follows. In section 2 we summarize the structure of  $N = 2$  super Yang-Mills theory and the results and predictions of [2] that are relevant to the problem at hand. In section 3 we discuss monopole dynamics in  $N = 2$  gauge theories

---

<sup>1</sup> The precise duality group involves a semi-direct product of  $SL(2, Z)$  and  $Spin(8)$  which is described in [2] and section 2 below.

emphasising the effects of the coupling to matter fermions. In particular we show that the low-energy monopole dynamics is determined by a supersymmetric quantum mechanics on the moduli space of BPS  $k$ -monopole configurations coupled to a natural  $O(k)$  connection that is constructed from the matter fermion zero modes. Our main results are in section 4 which contains an analysis of the BPS dyon spectrum in the sectors with magnetic charge one and two. The analysis in the charge two sector eventually reduces to a calculation of the index of the Dirac operator on the Atiyah-Hitchin manifold coupled to the Levi-Civita connection and an additional  $O(2)$  connection. We show that for magnetic charge one and two the dyon spectrum is in agreement with the conjectures of [1]. We also discuss the extension of these conjectures to higher monopole moduli spaces. The final section contains brief conclusions.

## 2. $N = 2$ Super Yang-Mills and the Seiberg-Witten Conjecture

### 2.1. $N = 2$ Without Matter

Pure  $N = 2$  super Yang-Mills theory with gauge group  $SU(2)$  involves a single vector supermultiplet consisting of a gauge field  $A_\mu$ , Weyl fermions  $\lambda, \psi$ , and a complex scalar  $\phi$ , all in the adjoint representation. The fields  $A_\mu, \lambda$  comprise an  $N = 1$  vector multiplet  $W_\alpha$  while  $\psi, \phi$  are the components of a  $N = 1$  chiral superfield  $\Phi$ . In component form the bosonic part of the Lagrangian is

$$S = \frac{1}{16\pi} \text{Im} \int \tau \text{Tr}(F \wedge F + i * F \wedge F) + \frac{1}{g^2} \int d^4x \left( (D_m \phi)^\dagger (D^m \phi) - [\phi, \phi^\dagger]^2 \right), \quad (2.1)$$

where  $\tau = \theta/2\pi + i4\pi/g^2$ .

### 2.2. Classical Theory

Classically the theory defined by (2.1) has a vacuum state for every gauge inequivalent minimum of the potential

$$V(\phi) = \frac{1}{g^2} \text{Tr}[\phi, \phi^\dagger]^2. \quad (2.2)$$

If we choose a gauge in which  $\phi = \frac{1}{2}a\sigma^3$ , the classical moduli space of vacua is the complex  $u$  plane with  $u = \frac{1}{2}a^2 = \text{Tr}\phi^2$ .

For each point with  $u \neq 0$  the perturbative spectrum of the theory consists of the massless photon and its superpartners and the massive  $W^\pm$  states and their superpartners. At the semi-classical level, the spectrum also includes monopole and dyon states with spin

0 and  $1/2$  with charges  $(n_m, n_e) = (1, n_e)$ , where the two integers  $n_m$  and  $n_e$  specify the electric and magnetic charges of the state, respectively [4,6]. After incorporating the corresponding antiparticles, these dyons fill out an  $N = 2$  hypermultiplet of the  $N = 2$  supersymmetry.

The masses of all states in this theory, including monopoles and dyons, satisfy a Bogomol'nyi bound

$$M \geq \sqrt{2}|Z| = \sqrt{2}|a(n_e + \tau n_m)|. \quad (2.3)$$

This bound is derived from the  $N = 2$  supersymmetry algebra in which  $Z$  appears as a central charge [7]. States that saturate the Bogomol'nyi bound are called BPS states and form short representations of the  $N = 2$  supersymmetry algebra.

The perturbative electrically charged states are all BPS states. For non-zero  $n_m$ , the Bogomol'nyi bound is saturated if and only if the classical monopole solutions obey the first order Bogomol'nyi equations

$$B^i = \pm D^i \phi. \quad (2.4)$$

Here the upper sign corresponds to monopoles ( $n_m > 0$ ) and the lower sign to anti-monopoles ( $n_m < 0$ ). The moduli space of solutions of these Bogomol'nyi equations is the starting point for determining the semiclassical existence of BPS monopole and dyon states, as we shall discuss later.

### 2.3. Quantum Theory

The quantum theory is much more complicated. We will not attempt to summarize the analysis of [1] but will just note the following points.

1. The quantum moduli space  $\mathcal{M}$  is also the  $u$ -plane but with singularities at  $u = \pm 1, \infty$ . Over the  $u$ -plane there is a flat  $SL(2, Z)$  bundle with specified monodromy.
2. The full  $SU(2)$  gauge symmetry is never restored on  $\mathcal{M}$ . As a result the theory should possess magnetic monopoles at all values of  $u$ .
3. Renormalization of the formula (2.3) and the monodromy are consistent with dyons of charge  $(n_m, n_e) = (1, n_e)$  becoming massless at the singularities at  $u = \pm 1$ .
4. Although there are jumping curves [1] across which BPS states can decay, it is possible to move in from weak coupling ( $u = \infty$ ) without crossing such a curve. As a result, the states which become massless at the singularities  $u = \pm 1$  must be visible as BPS states at weak coupling.

As we mentioned, and will be reviewed briefly below, the states  $(1, n_e)$  do exist as BPS states at weak coupling.

## 2.4. $N = 2$ With Matter Fields

Theories with  $N = 2$  supersymmetry can also contain hypermultiplets which consist of two Weyl fermions in conjugate representations of the gauge group and two complex scalars, also in conjugate representations. These fields are assembled into  $N = 1$  chiral superfields  $Q$  and  $\tilde{Q}$  transforming in conjugate representations. Following [2] we will consider  $N_f$  hypermultiplets in the fundamental representation of  $SU(2)$ ,  $Q^I$ ,  $\tilde{Q}_I$ ,  $I = 1, 2, \dots, N_f$ .

The terms in the Lagrangian involving the hypermultiplets consist of canonical kinetic energy terms as well as a coupling term given in  $N = 1$  superfield language by the superpotential

$$W = \sqrt{2} \sum_I \tilde{Q}_I \Phi Q^I. \quad (2.5)$$

The analysis of [2] relied on the possibility of adding mass terms for the hypermultiplets in (2.5). Here we will be content to analyze the theory in the limit of vanishing masses for these fields.

The global flavor symmetry group of (2.5) will be important in the later analysis. For general gauge group there is a  $SU(N_f) \times U(1)$  flavor symmetry which acts on the fields  $Q^I$ ,  $\tilde{Q}_I$  which transform as a  $N_f, \bar{N}_f$ . However for  $SU(2)$  the fundamental representation is pseudoreal rather than complex and as a result  $Q^I$  and  $\tilde{Q}_I$  lie in equivalent representations. This leads to a  $SO(2N_f)$  flavor symmetry which can be made evident through a change of basis:  $(Q^I, \tilde{Q}_I) \rightarrow Q'^i, i = 1, \dots, 2N_f$ . As we shall see, there exist monopoles and dyons transforming as spinors of  $SO(2N_f)$  and so more precisely the flavor symmetry is  $Spin(2N_f)$ .

## 2.5. Classical Theory

The classical moduli space of vacua of (2.5) is complicated by the possibility of the matter scalar fields acquiring vacuum expectation values. This leads to “Higgs branches” of the classical moduli space along which the gauge symmetry is completely broken. Since these branches do not have classical monopole solutions we will not consider them further. There is in addition a “Coulomb branch” along which the gauge symmetry is broken to  $U(1)$ . This branch can again be parametrized by the gauge invariant quantity  $u = \langle \text{Tr} \phi^2 \rangle$ . Just as in the pure  $N = 2$  case, on the Coulomb branch the masses of all states in the theory satisfy the Bogomol’nyi bound (2.3). The perturbative states saturate the bound and hence are BPS states. We will investigate in detail the existence of additional magnetically charged BPS states in later sections.

## 2.6. Quantum Theory

The structure of the quantum theory is now considerably more complicated but again involves an analysis of the monodromy of families of elliptic curves. For our purposes the main points are the following.

1. BPS states which become massless at the singularities in the  $u$  plane must again be visible as BPS states at weak coupling.
2. The singularities for  $N_f = 1, 2$  are consistent with dyons of charge  $(n_m, n_e) = (1, n_e)$  becoming massless.
3. The singularities for  $N_f = 3$  require a state  $(2, 1)$  that transforms as an  $SO(6)$  singlet to become massless. The existence of this BPS state at weak coupling is thus required for consistency of the analysis in [2].
4. The  $N_f = 4$  theory is a scale invariant theory with, for vanishing bare masses, no renormalization of the BPS mass formula (2.3). The quantum moduli space for this theory is thus the same as the classical moduli space. In particular the  $SU(2)$  symmetry is restored at the origin of the  $u$  plane. There is a conjectured exact  $S$ -duality which predicts the presence of a  $SL(2, Z)$  invariant dyon spectrum as discussed in the following subsection.

## 2.7. Predictions for the Dyon Spectrum

As mentioned above, analysis of the  $N_f = 3$  theory predicts the existence of a BPS state with  $(n_m, n_e) = (2, 1)$  at weak coupling. The  $N_f = 4$  theory gives rise to a richer set of predictions as a consequence of a conjectured exact  $S$ -duality of the spectrum. The precise duality group is conjectured to be the semi-direct product  $Spin(8) \ltimes SL(2, Z)$  [2]. The mod 2 reduction of  $SL(2, Z)$  is homomorphic to  $S_3$ , which is both the permutation group of three objects and the group of outer automorphisms of  $Spin(8)$ . Thus  $SL(2, Z)$  acts on  $Spin(8)$  via this homomorphism.

The  $SL(2, Z)$  action can be made more explicit as follows. Label the states by  $(n_m, n_e)_r$  where  $n_m, n_e$  are the magnetic and electric charges, respectively and  $r$  denotes its  $Spin(8)$  representation. The action of the  $SL(2, Z)$  matrix

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{2.6}$$

is then given by

$$\begin{aligned}
\tau &\rightarrow \frac{\alpha\tau + \beta}{\gamma\tau + \delta} \\
a &\rightarrow (\gamma\tau + \delta)^{-1}a \\
(n_m, n_e)_r &\rightarrow [(n_m, n_e)M^{-1}]_{r'}.
\end{aligned} \tag{2.7}$$

The representation  $r'$  is determined by triality. The vector ( $v$ ), spinor ( $s$ ) and conjugate spinor ( $c$ ) representations  $r$  are transformed via the  $SL(2, Z) \rightarrow S_3$  homomorphism. Explicitly, the mod 2 reduction of the  $SL(2, Z)$  matrix gives the following permutations:

$$\begin{aligned}
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &\rightarrow \begin{cases} v \rightarrow s \\ s \rightarrow v \\ c \rightarrow c \end{cases} \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} &\rightarrow \begin{cases} v \rightarrow c \\ s \rightarrow s \\ c \rightarrow v \end{cases} \quad etc.
\end{aligned} \tag{2.8}$$

Beginning with the hypermultiplet of states  $(0, 1)$  in the  $\mathbf{8}_v$  representation, the elementary quark multiplet, the  $SL(2, Z)$  action generates the orbit of states  $(p, q)$  with  $p$  and  $q$  relatively prime, with the  $Spin(8)$  representation determined by the mod 2 grading:

$$(0, 1) - \mathbf{8}_v; \quad (1, 0) - \mathbf{8}_s; \quad (1, 1) - \mathbf{8}_c. \tag{2.9}$$

Returning to the Bogomol'nyi bound (2.3) and using the triangle inequality we deduce that these states have mass strictly less than that of any possible decay products and hence must be stable. Thus, duality predicts that this orbit of states exist in these specific representations, that they saturate the Bogomol'nyi bound and that they form hypermultiplets of the underlying  $N = 2$  supersymmetry.

Let us now consider the vector multiplet  $(0, 2)$  containing the  $W$  boson which transforms as a singlet under  $Spin(8)$ . From the Bogomol'nyi bound we first note that the  $W$  boson is only neutrally stable to the decay into two quarks. Seiberg and Witten suggested that it is possible that the  $W$  bosons are in fact distinct states somewhat analogous to bound states at threshold in non-relativistic quantum mechanics. If this is the right interpretation, and we will in fact provide evidence that it is, self-duality predicts another orbit of states  $(2p, 2q)$  with  $p$  and  $q$  relatively prime, each at threshold. Like the  $W$  boson multiplet, these states should all transform as singlets under  $Spin(8)$  and fill out a vector multiplet.

Thus, starting with the known states, duality predicts an orbit of hypermultiplets  $(p, q)$  transforming in eight-dimensional representations of  $Spin(8)$  and possibly an orbit of vector multiplets  $(2p, 2q)$  transforming as singlets<sup>2</sup>. Since the quantum moduli space is the same as the classical moduli space, there are no jumping curves where BPS states can decay. Consequently, the predicted orbits should exist for all points in the moduli space and in particular for weak coupling where we can look for them using semi-classical techniques.

For monopole number 1, duality requires a tower of hypermultiplets  $(1, 2q)$  transforming as  $\mathbf{8}_s$  and another  $(1, 2q + 1)$  transforming as  $\mathbf{8}_c$ . For monopole number 2, a tower of hypermultiplets  $(2, 2q + 1)$  transforming as  $\mathbf{8}_v$  is required. In addition, if the interpretation of the  $W$  boson mentioned above is correct, a tower of vector multiplets  $(2, 2q)$  transforming as  $Spin(8)$  singlets is needed. We will provide evidence for the existence of all of these states in the ensuing sections.

### 3. Monopole Dynamics in the Moduli Space Approximation

At weak coupling, the dynamics of monopoles can be studied using semiclassical techniques. In this section we show how the low-energy dynamics and in particular the spectrum can be determined by analyzing a particular supersymmetric quantum mechanics.

#### 3.1. Monopole moduli space

Working in the  $A_0 = 0$  gauge, static monopole solutions to the classical equations of motion are given by solutions to the Bogomol'nyi equations (2.4). The monopole moduli space is defined as the set of gauge equivalence classes of solutions. For monopole charge 1 the moduli space is simply  $\mathcal{M}_1 = R^3 \times S^1$ ; the  $R^3$  corresponds to the location of the monopole in space and the phase  $S^1$  corresponds to electric charge: it arises from “large” gauge transformations on the monopole solution (this is the way dyons arise in the  $A_0 = 0$  gauge).

The general  $k$ -monopole moduli space,  $\mathcal{M}_k$ , is a  $4k$ -dimensional hyperKähler manifold. It is possible to separate the center of mass motion and the total electric charge and consequently there is an isometric decomposition

$$\mathcal{M}_k = R^3 \times \frac{S^1 \times \tilde{\mathcal{M}}_k^0}{Z_k}, \quad (3.1)$$

where  $\tilde{\mathcal{M}}_k^0$  is a  $(4k - 4)$ -dimensional, simply connected hyperKähler manifold.

---

<sup>2</sup> Note that, as in the  $N = 4$  case, the states  $(0, 0)$ , the photon multiplet, lie in a single  $SL(2, Z)$  orbit.



### 3.2. Fermion zero modes

Differentiating the most general BPS monopole solution with respect to the moduli gives the bosonic zero modes in the small fluctuations about the solution. With fermions present there are additional fermionic zero modes. Let us first consider the adjoint fermions. The Callias index theorem [8] implies that for monopole number  $k$  the two Weyl fermions in the adjoint will give rise to  $2k$  complex zero modes. Since the BPS monopole solutions saturate the Bogomol'nyi bound (2.3), the solutions break half of the supersymmetries. The broken supersymmetries acting on the bosonic solution lead to four fermion Goldstone modes. For a single monopole these are the only zero modes that come from the adjoint fermions. For higher monopole number, one can show that the bosonic and fermionic zero modes are paired by the unbroken supersymmetries [6]. Since the bosonic zero modes are simply tangent vectors of  $\mathcal{M}_k$ , the set of adjoint fermionic zero modes naturally leads to the tangent bundle. Using the language of [9] we can say that the index bundle of the gauge equivalence classes of Dirac operators in the adjoint representation parametrized by points on the moduli space is simply the tangent bundle.

Now let us consider the fermion zero modes arising from the matter fermions in the fundamental representation. The Callias index theorem states that for monopole number  $k$  there are  $k$  real zero modes for each fundamental Weyl fermion. These zero modes are not related to any bosonic zero modes by supersymmetry. For a single Weyl fermion Manton and Schroers discuss the index bundle of the Dirac operators in the fundamental representation and show that it is an  $O(k)$  bundle over the moduli space  $\mathcal{M}_k$  [9]. They denote it  $\text{Ind}_k$ . This bundle has a natural connection given by

$$A_a^{AB}(X) = \int d^3x \lambda^A(x, X)^\dagger \frac{\partial}{\partial X^a} \lambda^B(x, X), \quad (3.2)$$

where  $\lambda^A(x, X^a)$ ,  $A = 1, \dots, k$  are the zero modes around a monopole solution specified by the coordinates  $X^a$  on  $\mathcal{M}_k$ . The curvature of this connection is of type  $(1, 1)$  with respect to each of the three complex structures on  $\mathcal{M}_k$  and hence the curvature is anti-self-dual [10]. Note that the analogue of the connection (3.2) for the index bundle of the adjoint fermions is the Levi-Civita connection on  $\mathcal{M}_k$ . For the  $N = 2$  models we are interested in, we have  $N_f$  hypermultiplets and hence  $2N_f$  Weyl fermions in the fundamental representation. This gives rise to  $2kN_f$  fermion zero modes which leads to  $2N_f$  copies of the  $O(k)$  bundle.

### 3.3. Collective coordinate expansion

Let us first briefly summarize how the collective coordinate expansion works for pure  $N=2$  QCD [6]. As usual, for each zero mode one introduces a collective coordinate. For the bosonic zero modes these are just the coordinates on the moduli space, the moduli  $X^a$  themselves. For the  $2k$  complex fermionic zero modes arising from the adjoint fermions, one must introduce  $4k$  real Grassmann-odd collective coordinates  $\psi^a$ . We expect the low-energy dynamics to be dominated by the dynamics of the zero modes. We can encapsulate this in a low-energy ansatz for the fields by assuming that all time dependence is via the collective coordinates. Heuristically, we have

$$\begin{aligned} A_i(x, t) &= A_i(x, X^a(t)), & \Phi(x, t) &= \Phi(x, X^a(t)), \\ \psi(x, t) &= F(\delta_a A_i, \delta_a \Phi, \psi^a(t)). \end{aligned} \quad (3.3)$$

Here  $(\delta_a A_i, \delta_a \Phi)$ ,  $a = 1, \dots, 4k$ , are the bosonic zero modes and  $F$  is the functional determining the supersymmetric pairing between the bosonic and fermionic zero modes mentioned in the last subsection (the explicit form of (3.3) is given in [6]). Substituting this ansatz into the pure  $N = 2$  QCD Lagrangian and integrating over space then leads to an  $N = 2$  supersymmetric quantum mechanics on the moduli space of BPS monopoles:

$$S = \frac{1}{2} \int dt \mathcal{G}_{ab}(X) (\dot{X}^a \dot{X}^b + i \psi^a D_t \psi^b), \quad (3.4)$$

where  $\mathcal{G}_{ab}$  is the metric on the moduli space which arises from the kinetic energy terms in the field theory [11].

With the inclusion of hypermultiplets we must introduce more Grassmann-odd collective coordinates corresponding to the extra fermionic zero modes. The low energy ansatz will include the terms

$$\lambda^i(x, t) = \frac{1}{\sqrt{2}} \sum_A \rho^{iA}(t) \lambda^A(x, X^a(t)), \quad (3.5)$$

where  $\lambda^A(x, X^a)$ ,  $A = 1, \dots, k$  are the fermion zero modes introduced in (3.2) and  $\rho^{iA}(t)$ ,  $i = 1, \dots, 2N_f$  are the real Grassmann-odd collective coordinates. Substituting into the Lagrangian and integrating over space, yields for the kinetic term:

$$\begin{aligned} \int d^4x \bar{\lambda}^i \not{D} \lambda^i &\rightarrow \frac{1}{2} \int d^4x [\rho^{iA}(t) \lambda^A(x, X(t))^\dagger] \partial_t [\rho^{iB}(t) \lambda^B(x, X(t))] \\ &= \frac{1}{2} \int dt (\rho^{iA} \dot{\rho}^{iA} + \rho^{iA} A_a^{AB} \dot{X}^a \rho^{iB}) \\ &\equiv \frac{1}{2} \int dt \rho^{iA} \mathcal{D}_t \rho^{iA}, \end{aligned} \quad (3.6)$$

where in the first line we used the fact that  $\lambda^A$  are zero modes and in the second we have used (3.2) and have chosen a basis of zero modes satisfying

$$\int d^3x \lambda^A(x, X)^\dagger \lambda^B(x, X) = \delta^{AB}. \quad (3.7)$$

Thus by substituting the full low-energy ansatz (3.3) and (3.5) into the Lagrangian we are led to consider the following supersymmetric quantum mechanics:

$$S = \frac{1}{2} \int dt \left( \mathcal{G}_{ab} \left[ \dot{X}^a \dot{X}^b + i\psi^a D_t \psi^b \right] + i\rho^{iA} \mathcal{D}_t \rho^{iA} + \frac{1}{2} F_{ab}^{AB} \psi^a \psi^b \rho^{iA} \rho^{iB} \right). \quad (3.8)$$

While we have not provided a detailed derivation of (3.8), supersymmetry essentially dictates this result given the presence of the  $O(k)$  connection (3.2) in the kinetic term for the matter zero modes (3.6). Since the monopole solutions break half of the supersymmetry, the low energy quantum mechanics must be invariant under the supersymmetries arising from the unbroken supersymmetries of the field theory. Given that the number of real components in the unbroken supersymmetries is  $2 \times 4$ , we expect to have a quantum mechanics with four real parameters or  $N = 2$  supersymmetry. The action (3.8) automatically has  $N = 1/2$  supersymmetry [12] (although there is a mismatch between the number of bosons and fermions this action still admits supersymmetries that are non-linearly realized). Additional supersymmetries restrict the target. In particular, for the action to admit  $N = 2$  supersymmetry the moduli space must be hyperKähler and the field strength of the gauge connection must be  $(1, 1)$  with respect to each of the complex structures. Happily, both statements are true.

### 3.4. Supersymmetric Quantum Mechanics

To discuss the low-energy dynamics of the monopoles and particularly the spectrum of monopoles and dyons we need to quantize the action (3.8). The quantization of the action without the matter fermions  $\rho$  and its connection to the spectrum of BPS monopoles in the pure  $N = 2$  theory was described in detail in [13, 6]. The anti-commutation relations of the  $\psi^a$  imply that the states are either holomorphic forms or spinors on the moduli space. In fact on a hyperKähler manifold these two descriptions are equivalent. We will work with spinors.

The new ingredient in (3.8) is the presence of the  $\rho$  fields. Setting  $\hbar = 1$  their commutation relations are given by

$$\{\rho^{iA}, \rho^{jB}\} = \delta^{ij} \delta^{AB}. \quad (3.9)$$

The monopole states must provide a representation of this Clifford algebra. These representations can be decomposed under  $SO(2N_f) \times O(k)$  as will be described in the following section.

The four supersymmetry charges are given by

$$\begin{aligned} Q &= \psi^a \pi_a, & Q^{(m)} &= \psi^a J_a^{(m)b} \pi_b \\ \pi_a &= p_a - \frac{i}{4} \omega_{a\hat{b}\hat{c}} [\psi^{\hat{b}}, \psi^{\hat{c}}] - i A_a^{AB} \rho^{iA} \rho^{iB}, \end{aligned} \tag{3.10}$$

where  $J_a^{(m)b}$  are the three complex structures and the hatted indices are tangent space indices. Substituting  $p_a = -i\hbar\partial_a$  and  $\psi^{\hat{a}} = (2)^{-1/2}\gamma^{\hat{a}}$  we see that  $Q$  is just the Dirac operator acting on spinors in some representation of  $O(k)$ . The Hamiltonian is therefore just the square of this Dirac operator plus  $|a\tau k|$ . The additional constant term simply arises from a topological boundary contribution that exists in the field theory for solutions of the Bogomol'nyi equations.

The predictions we are aiming to test involve the existence of states saturating the Bogomol'nyi bound. Equivalently these are states that are annihilated by half of the supersymmetry charges of the  $N = 2$  field theory. In the moduli space approximation, this means we should look for states that are annihilated by the supersymmetry charges in (3.10). Moreover, they should have finite norm as is usual for bound states. Thus extra magnetically charge BPS states in the spectrum are in correspondence with the  $L^2$  kernel of the Dirac operator on  $\mathcal{M}_k$  coupled to the  $O(k)$  connection on the index bundle.

## 4. Dyon Spectrum

### 4.1. Magnetic Charge 1

As noted, the moduli space for a single monopoles is  $R^3 \times S^1$ . The supersymmetric quantum mechanics has a free Hamiltonian. As for ordinary monopoles, quantization of the bosonic coordinates on  $R^3 \times S^1$  leads to a spectrum of dyons  $(1, n_e)$  with continuous spacetime momentum, the quantized electric charge resulting from quantization of the  $S^1$  part of the moduli space. Quantization of the fermionic coordinates tangent to  $R^3 \times S^1$  was described in [4] and [6]. The states can be thought of as four-component spinors on  $R^3 \times S^1$  and correspond to four different states in the field theory with spin 0 and 1/2. These four states make up a short BPS multiplet (an irreducible multiplet of the  $N = 2$  supersymmetry algebra that saturates the Bogomol'nyi bound has four states [7]). If we

combine these states with similar states that come from quantizing the anti-monopoles we obtain a complete hypermultiplet of  $N = 2$  supersymmetry.

Let us now analyze the effect of the  $\rho$  zero modes. For  $k = 1$  we have the anti-commutation relations

$$\{\rho^i, \rho^j\} = \delta^{ij}. \quad (4.1)$$

As is well known, the representation of this Clifford algebra consists of the  $2^{N_f}$  dimensional spinor representation of  $SO(2N_f)$ . This representation is reducible and splits into two irreducible representations, both of dimension  $2^{N_f-1}$ , under projection by the chirality operator in  $SO(2N_f)$  which following [2] we denote by  $(-1)^H$ .

However this is not quite the end of the story. Even for  $k = 1$  there is still a non-trivial bundle structure. The  $O(1)$  connection on the index bundle is non-trivial over the  $S^1$  factor and leads to

$$\text{Ind}_1 = R^3 \times \text{Möb}, \quad (4.2)$$

where Möb is the Möbius bundle over  $S^1$ . Physically this arises because the gauge transformation which generates a  $2\pi$  rotation about the  $S^1$  factor acts as the non-trivial element of the center of  $SU(2)$  [14,2]. Since the  $N_f$  matter fermions transform in the fundamental representation of  $SU(2)$ , there must be a correlation between the  $U(1)$  charge (as measured by rotation about the  $S^1$  factor) and the  $SO(2N_f)$  chirality (as measured by  $(-1)^H$ ). Specifically, it is expressed by the constraint

$$e^{i\pi Q} = (-1)^H, \quad (4.3)$$

where  $Q$  is the charge operator. Thus we see that states with  $(-1)^H = 1$  have even electric charge while states with  $(-1)^H = -1$  carry odd electric charge.

Putting this together eg for the  $N_f = 4$  case, yields a tower of hypermultiplets  $(1, 2n_e)$  in the  $\mathbf{8}_s$  representation and another tower  $(1, 2n_e + 1)$  in the  $\mathbf{8}_c$  representation in agreement with  $Spin(8) \ltimes SL(2, Z)$  duality [2].

#### 4.2. Magnetic Charge 2

By factoring out the center of mass motion, the full two monopole moduli space  $M_2$  can be expressed as

$$M_2 = R^3 \times \left( \frac{S^1 \times \tilde{M}_2^0}{Z_2} \right) \quad (4.4)$$

where  $\tilde{M}_2^0$  is the four-dimensional Atiyah-Hitchin manifold. It has a two-fold cover denoted  $\tilde{M}_2$  which can be written as an isometric product

$$\tilde{M}_2 \cong R^3 \times S^1 \times \tilde{M}_2^0 \quad (4.5)$$

where  $\tilde{M}_2^0$  is the two-fold cover of the reduced two monopole moduli space  $M_2^0$ .  $SO(3)$  acts on  $M_2^0$  and has orbits which are either a  $RP_2$  (at the bolt) or  $SO(3)/D$  away from the bolt with  $D \cong Z_2 \times Z_2$  the subgroup of diagonal matrices in  $SO(3)$ . We denote the generator of the explicit  $Z_2$  that appears in (4.4) by  $I_3$  (see [15] for more details).

Let us first briefly recall the quantization of the supersymmetric quantum mechanics that arises when there are no  $N = 2$  matter multiplets i.e. (3.8) with no  $\rho$  [6]. As noted, the quantization of the  $\psi^a$  implies that the states are spinors on  $M_2$ . The structure of (4.4) implies that these spinors are tensor products of spinors on  $R^3 \times S^1$  with spinors on  $\tilde{M}_2^0$ . The discussion of the spinors on  $R^3 \times S^1$  is essentially as for the single monopole case: the dyon states  $(2, n_e)$  are in a short BPS multiplet with spin 0 and 1/2. Combining them with the anti-monopole states then fills out the spin content of a hypermultiplet. In the normalisations we are using, for the pure  $N = 2$  case the electric charge is always even. The effect of the  $Z_2$  that appears in (4.4) is that dyon states whose electric charge is (is not) a multiple of four are associated with spinors on  $\tilde{M}_2^0$  that are even (odd) under the action of the  $I_3$ .

The supersymmetry charge  $Q$  acts as the Dirac operator on the spinors. Consequently, the Hamiltonian is the sum of the free Hamiltonian on  $R^3 \times S^1$  and the square of the Dirac operator on  $\tilde{M}_2^0$  and the constant topological term. Thus, to find new BPS states in the spectrum we must look for zero energy states on  $\tilde{M}_2^0$  or equivalently zero modes of the Dirac operator. We will in fact see that there are no zero modes of the Dirac operator and consequently no extra BPS states for the pure  $N = 2$  case.

Now we consider the matter fermions. The index bundle  $\text{Ind}_2$  is a real two-dimensional vector bundle over  $M_2$  with structure group  $O(2)$  which is described in detail in [9]. There is an obstruction to obtaining an orientable bundle on the non-simply connected manifold  $(S^1 \times \tilde{M}_2^0)/Z_2$ . One obtains an orientable bundle  $\widetilde{\text{Ind}}_2$  by pulling  $\text{Ind}_2$  back to  $S^1 \times \tilde{M}_2^0$ . For now we work with the  $U(1)$  bundle  $\widetilde{\text{Ind}}_2$ . Replacing the real Grassmann parameters  $\rho^{iA}$  with complex parameters  $\rho^i$ , the anticommutation relations become those of annihilation and creation operators:

$$\{\rho^i, \rho^{j*}\} = \delta^{ij}. \quad (4.6)$$

The states of the supersymmetric quantum mechanics are now spinors on  $M_2$ ,  $|\Psi\rangle$ , on which the algebra (4.6) is realized. Starting with a state  $|\Psi\rangle$  satisfying  $\rho^i|\Psi\rangle = 0$  we can build up the  $\rho$  Fock space by acting with the  $\rho^{i*}$  in the usual manner. There is a correlation between the number of  $\rho^{i*}$ 's excited,  $N_\rho$ , and the  $U(1)$  charge carried by the corresponding spinor on  $M_2$ . To see this note that the supersymmetry charge (3.10) acting on these states takes the form

$$Q = \not{D} - i(N_\rho - N_0)A, \quad (4.7)$$

where  $A$  is the  $U(1)$  gauge connection and  $N_0$  is a normal ordering constant. This normal ordering constant can be fixed by a discrete charge conjugation symmetry: for each state with  $U(1)$  charge  $q$  there should be another state with  $U(1)$  charge  $-q$  [16]. More precisely, there is a discrete symmetry which combines parity and a “magnetic charge conjugation” described in [9] which changes the sign of the  $U(1)$  charge and also the electric charge but leaves the magnetic charge invariant. This fixes  $N_0$  to be  $N_0 = -N_f$ .

Since the  $\rho^i$  carry the  $\mathbf{2N_f}$  representation of  $SO(2N_f)$ , there is also a correlation between the  $SO(2N_f)$  representation carried by the state and  $N_\rho$  and consequently the  $U(1)$  charge  $q$ . For the cases of most interest,  $N_f = 4, 3$ , we display the explicit correlation between  $q$  and the  $SO(8)$ ,  $SO(6)$  representations in the following table.

Table 1

$q$	$SO(8)$ rep	$SO(6)$ rep.
$\pm 0$	<b>35</b> + <b>35</b>	<b>10</b> + $\overline{\mathbf{10}}$
$\pm 1$	<b>56</b>	<b>15</b>
$\pm 2$	<b>28</b>	<b>6</b>
$\pm 3$	<b>8<sub>v</sub></b>	<b>1</b>
$\pm 4$	<b>1</b>	

As in the pure  $N = 2$  case the spinors are tensor products of spinors on  $R^3 \times S^1$  and spinors on  $\tilde{M}_2^0$ . The states on  $R^3 \times S^1$  lead to spin 0 and spin 1/2. Extra BPS states arise from states that are annihilated by the Hamiltonian and consequently the supersymmetry charge. Thus we need to look for zero modes of the Dirac operator on  $\tilde{M}_2^0$  coupled to the  $U(1)$  connection which have finite  $L^2$  norm.

We begin by discussing some vanishing theorems. The square of the Dirac operator acting on a charge  $q$  field can be expressed as

$$\not{D}\not{D}\psi = D^2\psi + \frac{1}{8}R_{abcd}\gamma^{ab}\gamma^{cd}\psi + \frac{q^2}{2}F_{ab}\gamma^{ab}. \quad (4.8)$$

For the case of interest, the manifold is four-dimensional and both curvatures are anti-self-dual. Using this we have

$$\not{D}\not{D}\psi = D^2\psi + \frac{q^2}{2}F_{ab}\gamma^{ab}\frac{(1+\gamma_5)}{2}\psi, \quad (4.9)$$

where  $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$ .

First consider  $q = 0$ . Multiplying (4.9) by  $\psi^\dagger$  and integrating by parts we deduce that  $|\not{D}\psi|^2 = |D_a\psi|^2$ . Since the only covariantly constant spinor with finite  $L^2$  norm is zero, we deduce that there are no nontrivial zero modes of the Dirac operator with finite norm. This means that there are no extra BPS states in the spectrum of the  $N = 2$  theory without matter. In fact this is also a necessary requirement for the  $N = 4$  theory to be self-dual. The reason is that a harmonic spinor on  $\tilde{M}_2^0$  is equivalent to a harmonic holomorphic differential form on  $\tilde{M}_2^0$  and consequently a BPS state in the  $N = 4$  theory. However, Sen showed that  $S$ -duality predicts that only a single anti-self-dual harmonic form (which cannot be holomorphic) should exist on  $\tilde{M}_2^0$ .

For general  $q$ , by the same reasoning, (4.9) implies that the only non-trivial Dirac zero modes with finite norm satisfy  $\gamma_5\psi = \psi$ . To calculate the number of these we use the Atiyah-Patodi-Singer index theorem. The index theorem for manifolds with boundary reads in this case

$$Index(\not{D}) = \frac{1}{192\pi^2} \int_M \text{Tr} R \wedge R - \frac{q^2}{8\pi^2} \int_M F \wedge F + \int_{\partial M} Q + \frac{1}{2}[\eta(0) + h_D], \quad (4.10)$$

where  $q$  is the charge of the fermion under the  $U(1) \subset O(2)$  part of the connection. Here  $Q$  involves Chern-Simons-like contributions as described e.g. in [17].  $\eta(0)$  is the Dirac  $\eta$ -invariant of Atiyah-Patodi-Singer and  $h_D$  is the number of harmonic spinors on the boundary. The index must be evaluated on a fixed finite boundary  $r_0$  and then one takes the limit  $r_0 \rightarrow \infty$ . The index then counts the difference in the number of harmonic spinors satisfying the Atiyah-Patodi-Singer boundary conditions (see for example [17]). As far as we are aware there is no known proof that these correspond to  $L^2$  boundary conditions, but in all cases that we know of, they appear to. In particular they do for Taub-NUT space and



since  $\tilde{M}_2^0$  asymptotically approaches Taub-NUT space exponentially in the radial distance (see below) we assume that they do for the Atiyah-Hitchin manifold.

Before evaluating the various terms in (4.10) we first give some more details about  $\tilde{M}_2^0$ . The explicit metric on  $\tilde{M}_2^0$  is known and is given by

$$ds^2 = f(r)^2 dr^2 + a(r)^2 (\sigma_1^R)^2 + b(r)^2 (\sigma_2^R)^2 + c(r)^2 (\sigma_3^R)^2. \quad (4.11)$$

Here the  $\sigma_i^R$  are left-invariant one-forms on  $SO(3) = S^3/Z_2$  and the explicit forms are given, in the conventions of appendix A, by

$$\begin{aligned} \sigma_1^R &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi \\ \sigma_2^R &= \cos \psi d\theta + \sin \psi \sin \theta d\phi \\ \sigma_3^R &= d\psi + \cos \theta d\phi \end{aligned} \quad (4.12)$$

with  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$ ,  $0 \leq \psi < 2\pi$ . The angles are further restricted under the identification of the discrete right isometry [15] (see appendix A)

$$(\phi, \theta, \psi) I_x = (\pi + \phi, \pi - \theta, -\psi). \quad (4.13)$$

Note that we can equivalently let the range of  $\psi$  be  $0 \leq \psi < 4\pi$  and then divide out by  $I_x$ .

We will follow [15] in choosing  $f(r) = -b(r)/r$ . The radial functions  $a(r)$ ,  $b(r)$  and  $c(r)$  are given explicitly in [18]. Here we only need the asymptotic forms. Near  $r = \pi$  they take the form

$$\begin{aligned} a(r) &= 2(r - \pi) \left\{ 1 - \frac{1}{4\pi}(r - \pi) \right\} + \dots \\ b(r) &= \pi \left\{ 1 + \frac{1}{2\pi}(r - \pi) \right\} + \dots \\ c(r) &= -\pi \left\{ 1 - \frac{1}{2\pi}(r - \pi) \right\} + \dots \end{aligned} \quad (4.14)$$

Introducing appropriate Euler angles, it can be shown that after the identification by  $I_x$  the metric is smooth near  $r = \pi$  and that  $r = \pi$  is an  $S^2$  or bolt [15]. Near infinity,  $r \rightarrow \infty$ , the functions take the form

$$\begin{aligned} a(r) &= r \left( 1 - \frac{2}{r} \right)^{1/2} + \dots \\ b(r) &= r \left( 1 - \frac{2}{r} \right)^{1/2} + \dots \\ c(r) &= -2 \left( 1 - \frac{2}{r} \right)^{-1/2} + \dots, \end{aligned} \quad (4.15)$$

where the neglected terms fall off exponentially with  $r$ . Thus the metric approaches Taub-NUT space with negative mass parameter. Since  $a = b$  asymptotically, in addition to the  $SO(3)$  isometries arising from the left action there is an extra  $U(1)$  isometry coming from right actions. Physically, it corresponds to the fact that the relative electric charge of two widely separated monopoles becomes a good quantum number. Due to the identifications on the Euler angles arising from  $I_x$ , the topology of the boundary is  $S^3/Z_4$ .

The curvature two-form of the  $O(2)$  connection on  $\widetilde{\text{Ind}}_2$  is anti-self-dual and  $SO(3)$  invariant and is given explicitly by [9]

$$F = \alpha(r) \left( d\sigma_1^R - \frac{fa}{bc} dr \wedge \sigma_1^R \right). \quad (4.16)$$

where  $\alpha(r)$  obeys the ordinary differential equation

$$\frac{d\alpha}{dr} = -\frac{fa}{bc}\alpha. \quad (4.17)$$

From physical arguments it is known that the first Chern number of  $\widetilde{\text{Ind}}_2$  is  $\pm 1$ . Following [9] we fix  $c_1 = -1$  for concreteness. This fixes the normalization of  $\alpha(r)$  so that  $\alpha(\pi) = 1/2$ . As an aside, note that this harmonic anti-self-dual two form is in fact the same two-form corresponding to a BPS bound state in the  $N = 4$  theory found by Sen [5].

Let us now evaluate (4.10) for  $\tilde{M}_2^0$ . We first consider the Chern-Simons terms. It has been explicitly shown that the Chern-Simons pieces vanish for Taub-NUT space [17]. Since the Atiyah-Hitchin metric asymptotically approaches Taub-NUT space and differs only by terms exponentially small in  $r$ , we conclude that the Chern-Simons terms make no contribution to the index. Next consider the volume terms. A straightforward calculation which we present in appendix B gives

$$\frac{1}{192\pi^2} \int_M \text{Tr} R \wedge R - \frac{q^2}{8\pi^2} \int_M F \wedge F = \frac{1}{6} - \frac{q^2}{8}. \quad (4.18)$$

Proceeding to the boundary contributions in (4.10), we need to evaluate  $\eta(0)$  and  $h_D$  on the  $S^3/Z_4$  boundary of  $\tilde{M}_2^0$ . Now the  $\eta$  invariant,  $\eta(0)$ , is defined to be the analytic continuation to  $s = 0$  of

$$\eta(s) = \sum_{\lambda \neq 0} \lambda^{-s} \text{sign} \lambda, \quad (4.19)$$

where the  $\lambda$  are the eigenvalues of the Dirac operator on the boundary coupled to the flat  $O(2)$  connection on  $\widetilde{\text{Ind}}_2$ . We present the details of the calculation in the appendix where

we also show that there are no harmonic spinors on the boundary and hence  $h_D = 0$ . We obtain

$$\eta_q(0) = \frac{2}{3} + \frac{1}{4}([q+2]_4^2 - 4[q+2]_4), \quad (4.20)$$

where  $[q+2]_4 = (q+2) \bmod 4$ . Combining this result with (4.18) we conclude that there are no zero modes with charge  $q = 0, \pm 1, \pm 2$ , one zero mode for  $q = \pm 3$  and two each for  $q = \pm 4$ .

Let us now check how this fits in with the duality conjectures. For  $N_f = 4$  we want a tower of states  $(2, 2n_e + 1)$  forming a hypermultiplet transforming as a  $\mathbf{8}_v$  of  $SO(8)$ . In addition we need another tower of states  $(2, 2n_e)$  transforming as  $SO(8)$  singlets and filling out a vector multiplet. For  $N_f = 3$  we want a single hypermultiplet of states  $(2, 1)$  transforming as a singlet of  $SO(6)$ .

First consider the hypermultiplets. Recall that the spin content of a hypermultiplet is  $S_z = (0, 0, 0, 0, \pm 1/2, \pm 1/2)$  where  $S_z$  is the component of spin along the  $z$  axis. We have noted that the spinor on  $R^3 \times S^1$  corresponds to four states in a short BPS multiplet with  $S_z = (0, 0, \pm 1/2)$ . If these spinors are combined with one zero mode of the Dirac operator on  $\tilde{M}_2^0$  with zero angular momentum we will obtain a BPS hypermultiplet after we also include the corresponding states that come from quantizing the zero modes around the charge 2 anti-monopole<sup>3</sup>. If in addition the zero mode of the Dirac operator has  $U(1)$  charge  $q = \pm 3$  then according to table 1 the hypermultiplet will transform as an  $\mathbf{8}_v$  for the  $N_f = 4$  case and as a singlet for the  $N_f = 3$  case.

Now consider the vector multiplets. The spin content of the vector multiplet is  $S_z = (0, 0, \pm 1/2, \pm 1/2, \pm 1)$ . To obtain this spin content we need to combine the spinor on  $R^3 \times S^1$  which has spin  $S_z = (0, 0, \pm 1/2)$  with *two* zero modes of the Dirac operator on  $\tilde{M}_2^0$ , one with  $S_z = 1/2$  the other with  $S_z = -1/2$ . To form a singlet representation of  $SO(8)$  the spinors on  $\tilde{M}_2^0$  must be zero modes of the Dirac operator with charge  $q = \pm 4$ .

Our analysis in terms of the index of the Dirac operator allows us to count states but for a detailed check of the duality predictions we also need to identify the angular momentum and electric charges of these states. We have not done this; to do so would require either a more sophisticated use of index theory (especially relating the indices on the bundles  $\text{Ind}_2$  and  $\widetilde{\text{Ind}}_2$ ) or what would be more desirable, an explicit construction of the zero modes of the Dirac operator. In spite of this it is possible on general grounds to almost completely determine what the spectrum must be. Consider first the charge  $q = \pm 3$

---

<sup>3</sup> Note that the moduli space for the anti-monopole is the same as that of the monopole.

zero modes. The constraint (4.3) implies that these states must carry odd electric charge. In addition, charge conjugation symmetry implies that they carry opposite electric charge. If they both carried arbitrary odd electric charge then we would get twice as many states as required by duality. However there is in fact a  $Z_4$  condition on the electric charges which follows from the analysis of [9]. The  $Z_2$  symmetry  $I_3$  acting on  $M_2$  squares to give a translation about the  $S^1$  factor. But since the holonomy about the  $S^1$  factor is  $-1$  (for odd  $U(1)$  charge) there is in fact a mod 4 condition on the charges. If the state at  $U(1)$  charge  $q = -3$  has electric charge  $+1 \pmod{4}$  then the state at charge  $q = +3$  must have electric charge  $-1 \pmod{4}$  by charge conjugation. Given this electric charge assignment and  $N = 2$  supersymmetry the only consistent possibility is that this state has angular momentum zero so that the total sets of states (after including the antiparticles) transform as hypermultiplets of  $N = 2$  supersymmetry. For  $N_f = 4$ , we see from table 1 that these states transform as  $\mathbf{8}_v$  of  $SO(8)$  in complete agreement with the predictions of [2]. For  $N_f = 3$  the states transform as singlets of  $SO(6)$  in agreement with the  $(2, 1)$  state predicted by [2].

Now consider the two states we found at  $q = \pm 4$ . By (4.3) these states must carry even electric charge. An assignment of charges and spins consistent with duality and charge conjugation symmetry (which, as mentioned in section 4.2, actually involves parity as well) and the presence of a mod 4 condition on the electric charges is the following. Assign the two  $q = 4$  states  $S_z = 1/2$ , and for one state electric charge  $0 \pmod{4}$  and the other charge  $2 \pmod{4}$ . For the two  $q = -4$  states assign  $S_z = -1/2$  and again electric charges  $0$  and  $2 \pmod{4}$ . This gives a spectrum of charged vector multiplets with a multiplicity consistent with duality. However, as far as we can see this assignment is not completely forced by consistency conditions and thus in principle a more refined test of duality would be provided by an explicit construction of the zero modes which would allow an unambiguous determination of these quantum numbers.

## 5. Magnetic Charge $k > 2$

In order to fully establish the  $SL(2, Z)$  duality of the spectrum of BPS states in the  $N = 2$ ,  $N_f = 4$  theory it is necessary to generalize the arguments in the previous section to higher magnetic charge. At present this seems difficult without a fuller understanding of the  $k$  monopole moduli space and particularly its asymptotic structure. There has been some recent progress in determining this structure for well-separated  $k$ -monopoles [19].

In the hope that this may eventually be better understood we will be content here to state the content of the duality conjecture for  $k > 2$ .

For magnetic charge  $k$  and  $N_f$  flavors the matter fermions have  $2kN_f$  zero modes in the monopole background and the low-energy effective action involves the coupling of the fermion zero modes  $\rho^{iA}$ ,  $A = 1, 2, \dots, k$ ,  $i = 1, 2, \dots, 2N_f$  to an  $O(k)$  connection on the  $k$  monopole moduli space  $\mathcal{M}_k$ . The anti-commutation relations of the  $\rho^{iA}$  give rise to a representation of the Clifford algebra associated to  $O(2kN_f)$  with the zero modes transforming as the fundamental  $2kN_f$ -dimensional representation which decomposes under

$$O(2kN_f) \rightarrow O(k) \times SO(2N_f) \quad (5.1)$$

as  $2kN_f \rightarrow (k, 2N_f)$ . The monopole ground state transforms as the  $2^{kN_f}$  dimensional spinor representation of  $O(2kN_f)$ .

Under the decomposition (5.1) we will have

$$2^{kN_f} \rightarrow \sum_i (r_k^i, r_{2N_f}^i). \quad (5.2)$$

The actual determination of the irreducible representations of  $O(k)$  that appear in (5.2) is somewhat complicated to state in general. The important point is that (5.2) gives a pairing between representations of  $O(k)$  and  $SO(2N_f)$ . Some general features of this pairing are immediate and in agreement with the general requirements of duality. For even  $k$  we can use the decomposition  $O(k) \rightarrow SU(k/2) \times U(1)$  to write the Clifford algebra in terms of creation and annihilation operators which transform as the  $2N_f$  of  $O(2N_f)$ . As a result the ground state will transform as a sum of tensor representations of  $O(2N_f)$  in agreement with the requirements of duality for  $N_f = 4$ . For odd  $k$ , on the other hand, we can use the decomposition  $SO(2N_f) \rightarrow SU(N_f) \times U(1)$  to write the fermion zero modes in terms of creation and annihilation operators transforming as  $(N_f, k) + (\bar{N}_f, k)$  which leads to ground states transforming as spinorial representations of  $O(2N_f)$  [20].

For  $N_f = 4$ ,  $S$ -duality requires an analysis of the index of the Dirac operator on the monopole moduli space  $\mathcal{M}_k$  coupled to the  $O(k)$  connection on the index bundle  $I_k$  through the representation of  $O(k)$  determined by the pairing in (5.2). In particular, as in the  $k = 2$  case analyzed in the last section, the index for even  $k$  should be one in the  $O(k)$  representation paired with the  $\mathbf{8}_v$  representation of  $O(8)$  (representing the  $SL(2, Z)$  duals of the quark hypermultiplets), should be two in the  $O(k)$  representation paired with the identity representation of  $O(8)$  (representing the  $SL(2, Z)$  duals of the gauge boson

states) and should vanish for all other representations of  $O(8)$ . For odd  $k$ , on the other hand, duality predicts index one for the representation of  $O(k)$  paired with the  $\mathbf{8}_s$  of  $O(8)$ , index one for the representation of  $O(k)$  paired with the  $\mathbf{8}_c$  of  $O(8)$  (corresponding to the  $SL(2, Z)$  duals of the hypermultiplets  $(1, 0)$  and  $(1, 1)$ , respectively) and vanishing index for all other representations. In addition the electric charge assignments and rotational quantum numbers for these states must be consistent with duality.

## 6. Conclusions

We have verified the predictions of Seiberg and Witten for the spectrum of dyon bound states in  $N = 2$ ,  $SU(2)$  super Yang-Mills theory coupled to  $N_f$  matter multiplets in the case of magnetic charge  $k = 2$ . The most dramatic result is the existence of a spectrum compatible with  $SL(2, Z)$  duality for the  $N_f = 4$  theory as predicted by Seiberg and Witten. This includes the existence of the  $SL(2, Z)$  duals of the gauge bosons as “bound states at threshold,” an interpretation suggested in [2] but not clearly required by duality.

It would be useful to extend the techniques used in this paper to further check the duality conjecture by giving a more precise determination of the electric charges of the dyon states and an explicit calculation of the angular momentum of the zero modes by explicit construction of the zero modes of the Dirac operator for monopole charge 2. Of most interest of course, would be to extend these results to higher magnetic charge by explicitly verifying the predictions made in section 5.

More generally, it would be very interesting to extend the duality analysis of the  $SU(2)$ ,  $N_f = 4$  case to other gauge groups which have a field content leading to finite  $N = 2$  theories.

The analysis we have presented here also makes clear how little is understood about duality. A prediction which is simple to state and follows naturally from an analysis of the dynamics of  $N = 2$  gauge theory can at present be verified only by detailed calculations involving intricate cancellations and then only for low values of the magnetic charge. Hopefully a deeper understanding of duality will allow us to understand the spectrum of dyon bound states without recourse to the sort of analysis presented here.

## Acknowledgements

We thank A. Dabholkar, F. Dowker, G. Gibbons, N. Hitchin, G. Moore, J. Preskill, N. Seiberg, A. Sen, A. Strominger, E. Witten and especially S. Dowker for helpful discussions. JPG is supported by the U.S. Dept. of Energy under Grant No. DE-FG03-92-ER40701 and JH by NSF Grant No. PHY 91-23780.

## Appendix A. Conventions

The conventions adopted here are essentially the same as in the [15] with some minor changes<sup>4</sup>. We parametrize the three sphere with Euler angles. The general  $SU(2)$  rotation matrix can be constructed as follows<sup>5</sup>

$$\begin{aligned} U(\phi, \theta, \psi) &= U_z(\phi)U_y(\theta)U_z(\psi) \\ &= \begin{pmatrix} \cos \frac{\theta}{2} e^{i \frac{(\psi+\phi)}{2}} & \sin \frac{\theta}{2} e^{-i \frac{(\psi-\phi)}{2}} \\ -\sin \frac{\theta}{2} e^{i \frac{(\psi-\phi)}{2}} & \cos \frac{\theta}{2} e^{-i \frac{(\psi+\phi)}{2}} \end{pmatrix}. \end{aligned} \quad (\text{A.1})$$

The ranges of the angles are  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$  and  $0 \leq \psi < 4\pi$ . The  $SO(3)$  group manifold is obtained by restricting the range of  $\psi$  to be  $0 \leq \psi < 2\pi$  and identifying  $\psi \sim \psi + 2\pi$ .

By expanding  $U^{-1}dU$  in the basis  $(\frac{i}{2}\tau^i)$  where  $(\tau^i)$  are the Pauli matrices, we can construct the left invariant or “right” one forms  $\sigma_i^R$ . Similarly, the right invariant or “left” one forms  $\sigma_i^L$  can be constructed from  $dUU^{-1}$ . We get

$$\begin{aligned} \sigma_1^R &= -\sin \psi d\theta + \cos \psi \sin \theta d\phi \\ \sigma_2^R &= \cos \psi d\theta + \sin \psi \sin \theta d\phi \\ \sigma_3^R &= d\psi + \cos \theta d\phi \end{aligned} \quad (\text{A.2})$$

and

$$\begin{aligned} \sigma_1^L &= \sin \phi d\theta - \cos \phi \sin \theta d\psi \\ \sigma_2^L &= \cos \phi d\theta + \sin \phi \sin \theta d\psi \\ \sigma_3^L &= d\phi + \cos \theta d\psi. \end{aligned} \quad (\text{A.3})$$

The superscript  $R$  ( $L$ ) refers to the fact that the left (right) invariant one forms are dual to left (right) invariant vector fields  $\xi_i^R$  ( $\xi_i^L$ ) which generate right (left) group actions. We will also refer to  $\xi_i^R$  as a right vector field and to  $\xi_i^L$  as a left vector field. The explicit form of the dual vector fields satisfying  $\langle \xi_i^R, \sigma_j^R \rangle = \delta_{ij}$  and  $\langle \xi_i^L, \sigma_j^L \rangle = \delta_{ij}$  are given by

$$\begin{aligned} \xi_1^R &= -\cot \theta \cos \psi \partial_\psi - \sin \psi \partial_\theta + \frac{\cos \psi}{\sin \theta} \partial_\phi \\ \xi_2^R &= -\cot \theta \sin \psi \partial_\psi + \cos \psi \partial_\theta + \frac{\sin \psi}{\sin \theta} \partial_\phi \\ \xi_3^R &= \partial_\psi \end{aligned} \quad (\text{A.4})$$

---

<sup>4</sup> They are the same as Fano and Racah [21] if we interchange  $\phi$  and  $\psi$  and if we replace their  $(m', m)$  with  $(m, s)$ . They are related to the conventions of Landau and Lifschitz [22] with the replacements  $(\alpha, \beta, \gamma) \rightarrow (\psi, \theta, \phi)$  and  $(m', m) \rightarrow (m, s)$ .

<sup>5</sup> Note that this element of  $SU(2)$  is denoted  $U(\psi, \theta, \phi)$  in [22].

and

$$\begin{aligned}
\xi_1^L &= -\frac{\cos \phi}{\sin \theta} \partial_\psi + \sin \phi \partial_\theta + \cot \theta \cos \phi \partial_\phi \\
\xi_2^L &= \frac{\sin \phi}{\sin \theta} \partial_\psi + \cos \phi \partial_\theta - \cot \theta \sin \phi \partial_\phi \\
\xi_3^L &= \partial_\phi.
\end{aligned} \tag{A.5}$$

The left and right invariant one forms satisfy the Maurer-Cartan equations

$$\begin{aligned}
d\sigma_i^R &= \frac{1}{2} \epsilon_{ijk} \sigma_j^R \wedge \sigma_k^R \\
d\sigma_i^L &= -\frac{1}{2} \epsilon_{ijk} \sigma_j^L \wedge \sigma_k^L.
\end{aligned} \tag{A.6}$$

The Lie brackets of the left and right vector fields are given by

$$\begin{aligned}
[\xi_i^R, \xi_j^R] &= -\epsilon_{ijk} \xi_k^R \\
[\xi_i^L, \xi_j^L] &= \epsilon_{ijk} \xi_k^L \\
[\xi_i^R, \xi_j^L] &= 0.
\end{aligned} \tag{A.7}$$

The last equation expresses the fact that the right (left) vector fields are left (right) invariant. Note that the right vector fields satisfy the same algebra as that of our Lie algebra basis  $(\frac{i}{2}\tau^i)$  whereas there is an extra minus sign in the algebra of the left vector fields. We define angular momentum operators

$$L_i^R = -i\xi_i^R, \quad L_i^L = i\xi_i^L \tag{A.8}$$

satisfying

$$[L_i, L_j] = i\epsilon_{ijk} L_k \tag{A.9}$$

for either superscript. We also use the combinations  $L_\pm = L_1 \pm iL_2$ .

Following [21] and [22] we introduce the Wigner functions<sup>6</sup>

$$D_{ms}^j(\phi, \theta, \psi) = e^{im\phi} d_{ms}^j(\theta) e^{is\psi} \tag{A.10}$$

satisfying

$$\begin{aligned}
L_\pm^R D_{ms}^j &= [j(j+1) - s(s \pm 1)]^{1/2} D_{ms \pm 1}^j \\
L_3^R D_{ms}^j &= s D_{ms}^j
\end{aligned} \tag{A.11}$$

---

<sup>6</sup> This is denoted  $D_{ms}^j(\psi, \theta, \phi)$  in [22].



and

$$\begin{aligned} L_{\pm}^L D_{ms}^j &= -[j(j+1) - m(m \mp 1)]^{1/2} D_{m \mp 1s}^j \\ L_3^L D_{ms}^j &= -m D_{ms}^j \end{aligned} \quad (\text{A.12})$$

Note that  $D_{ms}^{1/2}(\phi, \theta, \psi) = U(\phi, \theta, \psi)$ .

Next we analyze the discrete symmetries that appear in the discussion of the two monopole moduli space. The metric (4.11) is constructed from  $\sigma^R$  and hence is left invariant; the left vectors  $\xi_i^L$  are Killing vectors. By restricting the range of  $\psi$  to be  $0 \leq \psi < 2\pi$  these generate the isometry group  $SO(3)$ . Additional isometries come from right actions. Following [15] we consider right actions corresponding to rotations of  $\pi$  about the  $x, y$  and  $z$  axes. These  $SU(2)$  matrices, consistent with (A.1), are

$$U_x(\pi) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad U_y(\pi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad U_z(\pi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad (\text{A.13})$$

Calculating  $U(\phi, \theta, \psi)U_x(\pi) = U(\phi', \theta', \psi')$  etc. we are led to the following transformations  $I_x$  etc. on the angles:

$$\begin{aligned} (\phi, \theta, \psi)I_x &= (\pi + \phi, \pi - \theta, -\psi) \\ (\phi, \theta, \psi)I_y &= (\pi + \phi, \pi - \theta, \pi - \psi) \\ (\phi, \theta, \psi)I_z &= (\phi, \theta, \pi + \psi). \end{aligned} \quad (\text{A.14})$$

These transformations each change the sign of two of the left invariant one forms and hence leave the metric invariant confirming that they are indeed discrete isometries of the left invariant metric. We have chosen the notation in (A.14) to emphasize that these are right actions. Using this notation we have [15]

$$I_x I_z = I_y. \quad (\text{A.15})$$

Note also that  $I_x^2 = I_y^2 = I_z^2$  and is the antipodal map on  $S^3$ . Thus, in the definition of the Atiyah-Hitchin manifold one can begin with left invariant forms on the  $SO(3)$  group manifold and then divide out by  $I_x$  or equivalently one can start with  $SU(2)$  and divide out by  $I_x$  (which generates a  $Z_4$  in  $SU(2)$ ). Note the transformation  $I_3$  (the  $Z_2$  that appears in (4.4)) is simply  $I_z$  plus an additional action on the coordinate of the  $S^1$  (see [15]).

Using the formulae in appendix D of [21] or equivalently in section 58 of [22] we have  $d_{ms}^j(\pi - \theta) = (-1)^{j-m} d_{m-s}^j(\theta)$  and hence

$$\begin{aligned} D_{ms}^j(gI_x) &= (-1)^j D_{m-s}^j(g) \\ D_{ms}^j(gI_y) &= (-1)^{j+s} D_{m-s}^j(g) \\ D_{ms}^j(gI_z) &= (-1)^s D_{ms}^j(g), \end{aligned} \quad (\text{A.16})$$

where  $g = (\phi, \theta, \psi)$ . Note that the first of these differs from [15].

## Appendix B. Volume Contributions to Index Theorem

For the general metric (4.11) on the Atiyah-Hitchin manifold  $\tilde{M}_2^0$  we choose a vierbein

$$e^r = f(r)dr, \quad e^1 = a(r)\sigma_1^R, \quad e^2 = b(r)\sigma_2^R, \quad e^3 = c(r)\sigma_3^R. \quad (\text{B.1})$$

We will use  $\mu = e^r \wedge e^1 \wedge e^2 \wedge e^3$  as defining a positive orientation. Note that

$$\begin{aligned} \mu &= fabc \sin \theta d\theta \wedge d\phi \wedge d\psi \wedge dr \\ &= \sqrt{g} d\theta \wedge d\phi \wedge d\psi \wedge dr. \end{aligned} \quad (\text{B.2})$$

since  $fabc > 0$ . The spin connection is given by

$$\begin{aligned} \omega^{12} &= (1 + c'/f)\sigma_3^R, & \omega^{31} &= (1 + b'/f)\sigma_2^R, & \omega^{23} &= (1 + a'/f)\sigma_1^R, \\ \omega^{r1} &= -(a'/f)\sigma_1^R, & \omega^{r2} &= -(b'/f)\sigma_2^R, & \omega^{r3} &= -(c'/f)\sigma_3^R. \end{aligned} \quad (\text{B.3})$$

where we have used

$$\frac{a'}{f} = \frac{(b-c)^2 - a^2}{2bc} \quad \text{and} \quad \text{cyclic} \quad (\text{B.4})$$

following from anti-self-duality and the fact that the  $SO(3)$  action rotates the complex structures (see [18]).

The gravitational volume contribution to the index is

$$I_V^R = \frac{1}{192\pi^2} \int_{\tilde{M}_2^0} \text{Tr} R \wedge R = -\frac{1}{48\pi^2} \int_{\tilde{M}_2^0} (R^{r1} \wedge R^{r1} + R^{r2} \wedge R^{r2} + R^{r3} \wedge R^{r3}), \quad (\text{B.5})$$

where the anti-self-duality of the curvature ( $R^{r1} = -R^{23}$  etc.) has been used. In terms of  $\hat{a} = a'/f$ ,  $\hat{b} = b'/f$ , and  $\hat{c} = c'/f$  the curvature components are

$$\begin{aligned} R^{r1} &= -\hat{a}' dr \wedge \sigma_1^R + [-\hat{a} + \hat{b} + \hat{c} + 2\hat{b}\hat{c}] \sigma_2^R \wedge \sigma_3^R \\ R^{r2} &= -\hat{b}' dr \wedge \sigma_2^R + [\hat{a} - \hat{b} + \hat{c} + 2\hat{a}\hat{c}] \sigma_3^R \wedge \sigma_1^R \\ R^{r3} &= -\hat{c}' dr \wedge \sigma_3^R + [\hat{a} + \hat{b} - \hat{c} + 2\hat{a}\hat{b}] \sigma_1^R \wedge \sigma_2^R \end{aligned} \quad (\text{B.6})$$

which gives

$$\begin{aligned} I_V^R &= +\frac{1}{48\pi^2} \int_{\tilde{M}_2^0} G(r) \sigma_1^R \wedge \sigma_2^R \wedge \sigma_3^R \wedge dr \\ &= -\frac{1}{48\pi^2} \int \sin \theta d\theta d\phi d\psi \int_{\pi}^{\infty} G dr \end{aligned} \quad (\text{B.7})$$

with

$$G(r) = \left[ \hat{a}^2 + \hat{b}^2 + \hat{c}^2 - 2\hat{a}\hat{b} - 2\hat{b}\hat{c} - 2\hat{a}\hat{c} - 4\hat{a}\hat{b}\hat{c} \right]'. \quad (\text{B.8})$$

Using the asymptotic forms for  $\hat{a}$ ,  $\hat{b}$ , and  $\hat{c}$  obtained from (4.14) and (4.15) we find

$$I_V^R = \frac{1}{24\pi^2} \frac{1}{2} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \int_0^{2\pi} d\psi = \frac{1}{6}. \quad (\text{B.9})$$

The factor of  $1/2$  arises from the  $I_x$  identification. Thus the gravitational volume contribution to the Dirac index is  $+1/6$ . As a check we can now calculate the Euler number:

$$\chi = -\frac{1}{16\pi^2} \int_{\tilde{M}_2^0} \text{Tr} R \wedge *R = 2, \quad (\text{B.10})$$

where we have used the anti-self-duality condition. This is consistent with the fact that  $\tilde{M}_2^0$  contracts onto a two sphere.

For the volume gauge contribution, we have the field strength (4.16)

$$F = d\alpha \wedge \sigma_1^R + \alpha \sigma_2^R \wedge \sigma_3^R \quad (\text{B.11})$$

so that for a charge  $q$  field we have

$$I_v^F = -\frac{q^2}{8\pi^2} \int_{\tilde{M}_2^0} F \wedge F = -\frac{q^2}{8\pi^2} \int_{\tilde{M}_2^0} d(\alpha^2) \sigma_1^R \wedge \sigma_2^R \wedge \sigma_3^R = -\frac{q^2}{8}, \quad (\text{B.12})$$

where we have used  $\alpha(\pi) = 1/2$  and that  $\alpha$  falls off exponentially with  $r$  as can be deduced from (4.17). Thus the sum of the volume terms for a charge  $q$  fermion is

$$I_V^R + I_V^F = \frac{1}{6} - \frac{q^2}{8}. \quad (\text{B.13})$$

## Appendix C. $\eta$ invariants

In this appendix we discuss the computation of the  $\eta$  invariant,  $\eta(0)$ , for the boundary of the  $O(2)$  bundle over the Atiyah-Hitchin manifold.  $\eta(0)$  is defined to be the analytic continuation to  $s = 0$  of

$$\eta(s) = \sum_{\lambda \neq 0} \lambda^{-s} \text{sign} \lambda. \quad (\text{C.1})$$

where the  $\lambda$  are the eigenvalues of the Dirac operator on the boundary of the manifold coupled to the flat  $O(2)$  connection. At infinity we have  $a \approx b \approx r$  and  $c \approx -2$  and so the boundary metric is

$$ds^2 = r^2((\sigma_1^R)^2 + (\sigma_2^R)^2) + 4(\sigma_3^R)^2. \quad (\text{C.2})$$

For the moment we ignore the identification by  $I_x$  and let  $\psi$  run from 0 to  $4\pi$  so that (C.2) defines a left-invariant (but not round) metric on  $S^3$ . Moreover, we also ignore the flat  $O(2)$  connection for the moment.

$\eta(0)$  is invariant under conformal rescalings of the metric so in order to compare with [23] we will consider the metric

$$ds^2 = \frac{1}{4}((\sigma_1^R)^2 + (\sigma_2^R)^2) + \frac{\mu^2}{4}(\sigma_3^R)^2 \quad (\text{C.3})$$

in the limit  $\mu \rightarrow 0$ . We then have the dreibein

$$e^1 = \frac{1}{2}\sigma_1^R, \quad e^2 = \frac{1}{2}\sigma_2^R, \quad e^3 = \frac{\mu}{2}\sigma_3^R \quad (\text{C.4})$$

and spin connections

$$\omega^{12} = (1 - \mu^2/2)\sigma_3^R, \quad \omega^{31} = (\mu/2)\sigma_2^R, \quad \omega^{23} = (\mu/2)\sigma_1^R. \quad (\text{C.5})$$

The Dirac equation is

$$i\gamma^a E^\mu{}_a (\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab})\psi = \lambda\psi \quad (\text{C.6})$$

The  $E_a = E^\mu{}_a \partial_\mu$  are dual to the  $e^a$  and using (C.4) we have  $E_1 = 2\xi_1^R$ ,  $E_2 = 2\xi_2^R$ ,  $E_3 = 2\xi_3^R/\mu$  where the  $\xi_i^R$  were introduced in (A.4). Substituting (A.8), the Dirac equation reads

$$-2 \begin{pmatrix} \mu^{-1}L_3^R & L_+^R \\ L_+^R & -\mu^{-1}L_3^R \end{pmatrix} \psi - \frac{1}{2\mu}(\mu^2 + 2)\psi = \lambda\psi. \quad (\text{C.7})$$

The eigenfunction can now be constructed in terms of the Wigner function  $D_{m,s}^j(g)$  introduced in (A.10). Using (A.11), the eigenfunctions of the Dirac operator are of two types.

The first type is of the form

$$\psi_{m,s}^{0j} = \begin{pmatrix} aD_{m,s}^j \\ bD_{m,s+1}^j \end{pmatrix} \quad (\text{C.8})$$

with  $s = -j, \dots, j-1$ . In order to be an eigenfunction we must have

$$\frac{b}{a} = -\frac{2s+1}{2\mu X} \pm \frac{1}{2\mu X} \sqrt{(2s+1)^2 + 4\mu^2 X^2} \quad (\text{C.9})$$

with  $X^2 = j(j+1) - s(s+1)$ . Note that as a function of  $s$  at fixed  $j$  we have

$$\frac{b}{a}(s) = \frac{a}{b}(-s-1). \quad (\text{C.10})$$

The eigenvalues are

$$\lambda_{\pm} = -\frac{\mu}{2} \mp \frac{1}{\mu} \sqrt{(2s+1)^2 + 4\mu^2 X^2}. \quad (\text{C.11})$$

Since the  $\eta$  invariant is left unchanged by a rescaling of all of the eigenvalues, to evaluate it in the limit  $\mu \rightarrow 0$  we rescale by a factor of  $\mu$  and then let  $\mu \rightarrow 0$ . In this limit we get for the rescaled eigenvalues

$$\lambda_{\pm} = \mp(2s+1). \quad (\text{C.12})$$

Since these are symmetric between positive and negative eigenvalues they do not contribute to  $\eta(s)$  in the limit  $\mu \rightarrow 0$ .

The second type of eigenfunctions have the form

$$\psi_m^{+j} = \begin{pmatrix} D_{m,j}^j \\ 0 \end{pmatrix} \quad (\text{C.13})$$

and

$$\psi_m^{-j} = \begin{pmatrix} 0 \\ D_{m,-j}^j \end{pmatrix} \quad (\text{C.14})$$

with eigenvalues

$$\lambda_+ = \lambda_- = -\frac{\mu}{2} - \frac{1}{\mu}(2j+1). \quad (\text{C.15})$$

After the rescaling by  $\mu$  and taking the limit we get eigenvalues  $\lambda_{\pm} = -(2j+1)$ . These eigenvalues are all negative. So we get eigenvalues  $-(2j+1)$  with degeneracy  $2(2j+1)$  with the 2 from  $\psi_{\pm}$  and the  $(2j+1)$  from the possible  $m$  values. The  $\eta$  invariant is thus

$$\eta(s) = - \sum_j 2(2j+1)(2j+1)^{-s} \quad (\text{C.16})$$

and since  $j$  takes on half-integer and integer values this is equal to

$$\eta(s) = -2 \sum_{p=1}^{\infty} p^{-s+1} \equiv -2\zeta(s-1) \quad (\text{C.17})$$

and  $\eta(0) = -2\zeta(-1) = 1/6$ . Note that in the limit  $\mu \rightarrow 0$  there are no harmonic spinors and hence  $h_D = 0$ .

So far we have just redone the standard computation of the  $\eta$  invariant on the squashed  $S^3$  [24] in the limiting case  $\mu \rightarrow 0$ . In extending this computation to the boundary of the Atiyah-Hitchin manifold there are two complications which we must deal with. The first is that the boundary is not  $S^3$  but rather  $SO(3)/Z_2 \equiv S^3/Z_4$  because of the identification by  $I_x$ . As a result the boundary has different inequivalent spin structures. Since the

Atiyah-Hitchin manifold is simply connected and consequently has a unique spin structure only one of these spin structures can extend smoothly to the interior. Determining the spin structure at the boundary appears to be somewhat subtle, luckily we will be able to determine its effect by consistency requirements. The second complication arises from the  $O(2)$  connection. The field strength of this connection falls off exponentially leaving a flat  $O(2)$  connection at the boundary. The holonomy of this connection at the boundary has been computed in [9] with the result that charge  $q$  fermions pick up a phase of  $e^{\pm iq\pi/2}$ .

Now let us first consider spinors with  $q = 0$ . Identifying under the action of  $I_x$  leads to  $\pi_1$  of the boundary being  $Z_4$ . The different spin structures on the boundary can be specified by giving the phase picked up by the fermion after propagating between two points related by  $I_x$ . Since  $I_x$  is a right action it commutes with the isometries  $SO(3)_L$  and consequently the phases picked up by the fermions fields cannot depend on  $m$ . As in the previous calculation the eigenfunctions (C.8) have no spectral asymmetry in the limit  $\mu \rightarrow 0$  and thus do not contribute to  $\eta(0)$ . We now consider the eigenfunctions (C.13) and (C.14). For a given  $j$  a single eigenfunction picking up a phase  $e^{\frac{2\pi i r}{N}}$  (in our case  $N = 4$  since  $\pi_1$  of the boundary is  $Z_4$ ) contributes a term  $-f(r, N, -1)$  to the  $\eta$  invariant where we have introduced the function

$$f(r, N, s) = \sum_{n=r \bmod N}^{\infty} n^{-s} \quad (\text{C.18})$$

with  $r$  and  $N$  integers. We have

$$f(r, N, s) = \sum_{n=0}^{\infty} (nN + r)^{-s} = N^{-s} \sum_{n=0}^{\infty} (n + r/N)^{-s} = N^{-s} \zeta(s, r/N) \quad (\text{C.19})$$

in terms of the generalized Riemann zeta function. The value we need may be evaluated with the result [25]

$$\begin{aligned} f(r, N, -1) &= N\zeta(-1, r/N) = -N \frac{B'_3(r/N)}{6} \\ &= -N \left( \frac{1}{12} + \frac{1}{2}((r/N)^2 - (r/N)) \right) \end{aligned} \quad (\text{C.20})$$

(it is amusing to note that the same sum arises in the computation of the dimension of twist fields for orbifold compactifications of string theory [26]). Assuming in general that the two eigenfunctions (C.13) and (C.14) pick up phases  $r_1$  and  $r_2$  due to the spin structure at infinity, they will give a contribution to the  $\eta$  invariant of  $-f(r_1, 4, -1) - f(r_2, 4, -1)$ .

Noting the values  $f(0, 4, -1) = -1/3$ ,  $f(1, 4, -1) = f(3, 4, -1) = 1/24$  and  $f(3, 4, -1) = 1/6$  we see that the only possibility which is consistent with an integer index and the vanishing theorem for  $q = 0$  mentioned in the text is  $r_1 = r_2 = 2$ .

Given this indirect evaluation of the effect of the spin structure for  $q = 0$  it is straightforward to generalize the calculation to non-zero  $q$ . Using the fact that the holonomy is  $e^{\pm\pi i q/2}$  there are two possibilities: either the two eigenfunctions pick up the same phase or opposite phases. The first possibility leads to the  $\eta$  invariant:  $-2f([q+2]_4, 4, -1)$ , where  $[q+2]_4$  is  $(q+2) \bmod 4$  (since the phase is  $\pi i(q+2)/2$ ). The second possibility leads to the  $\eta$  invariant:  $-f(q+2, 4, -1) - f(2-q, 4, -1)$  but these two expressions are equal. We thus conclude that the  $\eta$  invariant is given by

$$\eta_q(0) = \frac{2}{3} + \frac{1}{4}([q+2]_4^2 - 4[q+2]_4). \quad (\text{C.21})$$

Noting that  $h_D$  still vanishes, this then leads to the results stated in the text.

## References

- [1] N. Seiberg and E. Witten, Nucl. Phys. **B426** (1994) 19; ERRATUM-ibid. **B430** (1994) 485; hep-th/9407087.
- [2] N. Seiberg and E. Witten, Nucl. Phys. **B431** (1994) 484; hep-th/9408099.
- [3] C. Montonen and D. Olive, Phys. Lett. **72B** (1977) 117; P. Goddard, J. Nuyts and D. Olive Nucl. Phys. **B125** (1977) 1.
- [4] H. Osborn, Phys. Lett. **83B** (1979) 321.
- [5] A. Sen, Phys. Lett. **329** (1994) 217.
- [6] J. P. Gauntlett, Nucl. Phys. **B411** (1994) 443.
- [7] E. Witten and D. Olive, Phys. Lett. **78B** (1978) 97.
- [8] C. Callias, Comm. Math. Phys. **62** (1978) 213.
- [9] N. S. Manton and B. Schroers, Ann. Phys. **225** (1993) 290.
- [10] N. Hitchin as quoted in [9].
- [11] N. S. Manton, Phys. Lett. **B110** (1982) 54.
- [12] L. Alvarez-Gaume, J.Phys. **A16** (1983) 4177.
- [13] J. P. Gauntlett, Nucl. Phys. **B400** (1993) 103.
- [14] E. Witten, Phys. Lett. **B86** (1979) 283.
- [15] G. W. Gibbons and N. Manton, Nucl. Phys. **B274** (1986) 183.
- [16] R. Jackiw and C. Rebbi, Phys. Rev. **D13** (1976) 3398.
- [17] T. Eguchi, P. B. Gilkey, A. J. Hanson, Phys. Rept. **66** (1980) 213.
- [18] M.F. Atiyah and N. J. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, Princeton University Press, 1988.
- [19] G. Gibbons and N. Manton, “The Moduli space for well separated BPS monopoles,” hep-th/9506052.
- [20] See for example the discussion in Appendix 5.A of M. Green, J. Schwarz and E. Witten, “Superstring Theory I, ” (Cambridge University Press 1987 ).
- [21] U. Fano and G. Racah, Irreducible Tensorial Sets, Academic Press, (1959).
- [22] L. D. Landau and E. M. Lifshitz, Quantum Mechanics, Pergamon, (1977).
- [23] C. N. Pope, J. Phys. A: Math. Gen. **14** (1981) L133
- [24] N. Hitchin, Adv. Math. **41** (1974) 1.
- [25] I. S. Gradshteyn and I.M. Ryzhik, “Tables of Integrals, Series and Products,” (Academic Press, New York, 1965.)
- [26] L. Dixon, J. A. Harvey, C. Vafa and E. Witten, Nucl. Phys. **B261** (1985) 678; Nucl. Phys. **B274** (1986) 285.